SCCC 411B/MATH 523 Worksheet #1
Unconstrained population growth

1. In many cases, the rate of change of a variable quantity is proportional to the quantity itself. Consider human population. If a city of 100,000 is increasing at a rate of 1,500 persons/year, we would expect a similar city of 200,000 to be increasing at a rate of 3,000 persons/year and a city of 600,000 to be increasing at a rate of what?

   (a) For the cities described above, if P is the population at time t, and the net ***growth rate*** P' is proportional to P, compute the proportionality constant, which is customarily denoted by the letter k, in chemistry and mathematics, and by the letter r in biology. That is if $P' = rP$, determine the value of r. What are the units for r? This constant is generally called the ***per capita growth rate*** or the ***inartistic growth rate***. Why is the term "per capita" appropriate? (You may be tempted to think of P' as the derivative of P — and it is — but for the moment this is not as useful as thinking of it as the net growth rate.)

   (b) In 1985 the per capita growth rate of Poland was 9 persons/year per 1,000 persons. Fore Afghanistan it was 21.6 persons/year per 1,000 persons. If P denotes the population of Poland and A the population of Afghanistan, write the rate equations for P' and A' that govern the growth of these two populations.

   (c) In 1985 the population of of Poland was 37.5 million, and that of Afghanistan 15 million. What were the net growth rates in 1985? It it true that a population with a larger per capita growth rate also has a larger net growth rate? What is the correct conclusion that one can draw in comparing populations with different per capita growth rates?

   (d) If we use Δt to denote a time interval, and ΔP the change of the population of Poland during this time interval, write an equation that shows has the quantities ΔP, P' and Δt are connected. (HINT: It may help to consider the units on these quantities. Also here by "equation" we really mean an approximation.) Use this equation to compute the population of Poland in 1990, based on the available information.

   (e) Critique the solution you gave in part d. Can you see a way to get a better estimate for the population in 1990? (HINT: What happens to P' as soon as P changes?) Use your improved method, say with 10 steps, to track P from 1985 to 1990. Do the same of A. Sketch the graphs illustrating the population growth. This method is credit to Euler, the graphs that you obtained are ***piecewise linear*** (that is they should consist of set of line segments) and are called ***Euler polygons***.

The analysis that you made above can be put into a more familiar setting. Recall that the definition of the derivative (that is the rate of change) is $P' = \lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t}$. To if Δt is small P' is approximately equal to $\frac{\Delta P}{\Delta t}$. This can be expressed in the much more useful form $\boxed{\Delta P \approx P'\Delta t}$. What you have done above is apply this approximation over and over again: an initial value of P gives an initial value of $P' = rP$, which gives ΔP, which enables use to update the value of P, which gives a new P', and on and on it goes.

2. The "rate equation" $y' = ky$ governs the growth of the dependent variable y as function of the independent variable, let's say (to give it a name) t. At first we thought of y' standing for the "rate of change" of y, but as hinted at above: y' is the derivative of y with respect to the independent variable t and our equation is actually a differential equation.

   (a) Find the analytic solution of this differential equation: (HINT: rewrite $y' = ky$ as $\frac{dy}{dt} = ky$, separate variables, integrate both sides, consolidate constants, and do some algebra to solve for y. The constant of integration can be expressed in terms of y_0, the value of y at $t = 0$. or A different approach is to rewrite as $y' - ky = 0$, multiple by the "integrating factor" e^{-kt} to get $e^{-kt}y' - ke^{-kt}y = 0$. By the product rule this is equivalent to $\left(e^{-kt}y\right)' = 0$. Therefore $e^{-kt}y$ is a constant.)

   (b) Apply your formula to the cases of Poland and Afghanistan. What does the formula predict for the populations in 1990? How long will it take for each population to double in size?

3. Despite its flaws, this models of unconstrained population growth valid for populations over limited time spans in which case resources are for all practical purposes unlimited. Here's a sample textbook problems dealing with bacterial growth. A biology student finds a large glass bottle in which case she places a bacterial culture that doubles in size every minute. She notices the time is 2 p.m. At 3 p.m. the stuff complete fills the bottle.

   (a) At what time was the bottle half full? (This can be done without doing any calculations.)

   (b) For how much of the time between 2 p.m. and 3 p.m. was the bottle less than 1% full?

4. Soon we will lean how to use Maple, but you might want to think about this problems a little before then. The estimated population of the world (in millions) is given in the table below.

Year	1650	1750	1800	1850	1900	1920	1930	1940	1950	1960	1970	1975
Pop.	508	711	912	1131	1590	1811	2015	2249	2509	3008	3610	3967

   (a) Plot the population against time. Does this appear to be exponential growth?

   (b) Plot ln(Pop.) or $\log_{10}$(Pop.) against time. Do the points lie on a straight line?

   (c) How does the plot in part b help you decide whether the world population is growing exponentially or not?

   (d) If you can get hold of semi-log paper (i.e., logarithmic along the axis of the dependent variable) redo part a of this problem.