SCCC 411B/MATH 523 Worksheet #2 Fall 2005

A Brief Introduction to Vectors and Matrices

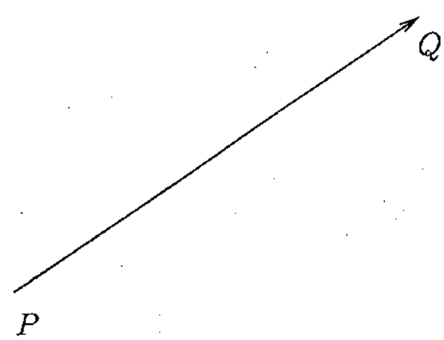
A scalar quantity is one can be described by single number. For example height, width, temperature, speed, mass, GPA, population size, and concentration. A vector quantity is one that needs two more more numbers to describe it. Examples are velocity (which involves both speed and direction), force (as it also involves direction), multi-vitamins (how much each of A, B, C, E, etc.), TV audience share (what % is watching each chanel), and statistics for animal of forest management (the number in each each age group for some a species of plant or animal). On this work sheet we will look at two dimensional vectors. That is ones that only depend on two numbers.

Our precise definition of a two dimensional ***vector*** is an ordered pair $\mathbf{v} = (a, b)$ of real numbers. Since boldface is not convenient for hand written work, one often writes $\vec{v}$, $\underset{\sim}{v}$, or $\underset{=}{v}$. We have special names for some vectors: $\mathbf{0} = (0, 0)$ (the ***zero vector***), $\mathbf{i} = (1, 0)$, $\mathbf{j} = (0, 1)$ (the ***unit vectors in the*** x ***and*** y ***directions***). This can be a little confusing at first as the ordered pair (a, b) can be either the coordinates of a points, or a vector. Hopefully it will generally be clear from context which is meant. (Some authors avoid this confusion by using either the notation $[a, b]$ or $\langle a, b\rangle$ for the vectors to distinguish it from (a, b).)

Let P and Q be points in the plan with coordinates (x, y) and (x', y') respectively. One of the most basic example of a vector is the ***displacement vector*** from P to Q:

$$\overrightarrow{PQ} = Q - P = (xx' - x, y' - y).$$

This is denoted geometrically as a arrow from P to Q. The ***length*** or ***magnitude*** of this



vector is $\mid \overrightarrow{PQ} \mid = \sqrt{(x' - x)^2 + (y' - y)^2}$. Vectors are ***added*** component wise. That is if $\mathbf{v} = (a, b)$ and $\mathbf{w} = (c, d)$, then $\mathbf{v} + vbw = (a + c, b + d)$. Likewise $\mathbf{v} - vbw = (a - c, b - d)$. If λ is a scalar, then the ***scalar multiplication*** of $\mathbf{v} = (a, b)$ by λ is $\lambda\mathbf{v} = (\lambda a, \lambda b)$. The ***dot product*** or ***scalar product*** of $\mathbf{v}$ and $\mathbf{w}$ is $\mathbf{v} \cdot \mathbf{w} = ac + bd$. Note that the dot product of two vectors is a scalar.

Exercise

1. Let $\mathbf{v} = (3, 4)$, $\mathbf{w} = (-2, 6)$, and $\mathbf{w} = (-1, 3)$. Compute:

(a) $\mathbf{v}+\mathbf{w}$, $\mathbf{v}-\mathbf{w}$, $3\mathbf{v}-5\mathbf{w}$.

(b) Scalars a and b such that $\mathbf{v} = a\mathbf{i}+b\mathbf{j}$. In general if $\mathbf{r} = (r_1, r_2)$ express $\mathbf{r}$ as $a\mathbf{i}+b\mathbf{j}$.

(c) a vector with twice the length of $\mathbf{v}$, but pointing in the opposite direction.

(d) Scalars r and s such that $\mathbf{w} = r\mathbf{v} = s\mathbf{u}$. Illustrate with a sketch of all the vectors involved.

(e) $\mathbf{u}\cdot\mathbf{w}$, $\mathbf{u}\cdot\mathbf{w}$, and $\mathbf{v}\cdot\mathbf{w}$.

Transformations. A ***transformation*** T of the plane $\mathbf{R}^2$, is a function that takes as inputs point (x, y) from the plane, and outputs a another point (u, v). This can be expressed as

$$T(x,y) = (u(x,y), v(x,y))$$

where $u(x, y)$ and $v(x, y)$ are scalar valued functions on the plane. Transformations of the simple form

$$T(x,y) = (ax+by, cx+dy)$$

where a, b, c, d are constants are ***linear transformations***. Fortunately linear transformations are relatively simple to analyze, and also commonly occur in applications. As an example consider a counterclockwise rotation about the origin by 90°. This is given by $T(x, y) = (-y, x)$ which is linear. Usually most convenient method for dealing with linear transformations is by matrices. Write vectors in $\mathbf{R}^2$ as column vectors. That is in the form

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

rather than (x, y). The 2×2 ***matrix*** entries a, b, c, d is

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

We multiple the vector $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ by the matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ by rule

$$A\mathbf{v} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax+by \\ cx+dy \end{bmatrix}.$$

We can therefore describe a linear transformation $T(x, y) = (ax + by, cx + dy)$ by giving its matrix. There is an easy way to find the matrix A of the linear transformation $T = (ax + by, cx + dy)$. Note that $\mathbf{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Thus, writing vectors as columns, we have

$$T(\mathbf{i}) = \begin{bmatrix} a \\ c \end{bmatrix}, \qquad T(\mathbf{j}) = \begin{bmatrix} b \\ d \end{bmatrix}.$$

Therefore

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = [T(\mathbf{i}), T(\mathbf{j})].$$

That is

Useful fact: *The matrix of the linear transformation T is the matrix A whose columns are $T(\mathbf{i})$ and $T(\mathbf{j})$. That is $A = [T(\mathbf{i}), T(\mathbf{j})]$.*

For example when T is the rotation by 90° counterclockwise, we have

$$T(\mathbf{i}) = \mathbf{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad T(\mathbf{j}) = -\mathbf{i} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Thus the matrix of T is

$$A = [T(\mathbf{i}), T(\mathbf{j})] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

Exercises

2. In each case find the matrix that produces the desired geometric transformation.

   (a) Reflection across the x-axis,

   (b) Reflection across the y-axis,

   (c) Reflection across the line $y = x$,

   (d) Rotation by 60° counterclockwise around the origin,

   (e) Rotation by $\theta°$ counterclockwise around the origin,

   (f) Stretching by a factor of 4 in the x-direction, and compressing by a factor of 3 in the y-direction, and

   (g) No change in the y-direction, but a shear factor of 3 times the y-coordinate in the x-direction. (Points all stay at their original height above of below the x-axis, but the higher up they are the more they get moved sideways.)